

GEOMETRIC TRANSITIONS BETWEEN CALABI–YAU THREEFOLDS RELATED TO KUSTIN-MILLER UNPROJECTIONS

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ABSTRACT. We study Kustin-Miller unprojections between Calabi–Yau threefolds, or more precisely the geometric transitions they induce. We use them to connect many families of Calabi–Yau threefolds with Picard number one to the web of Calabi–Yau complete intersections. This result enables us to find explicit description of a few known families of Calabi–Yau threefolds in terms of equations. Moreover, we find two new examples of Calabi–Yau threefolds with Picard group of rank one, which are described by Pfaffian equations in weighted projective spaces.

1. INTRODUCTION

In this paper a Calabi–Yau threefold X is a smooth projective threefold with trivial canonical divisor and vanishing cohomology $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Such varieties are intensely investigated in particular because of the crucial role they play in superstring theory. The simplest example of such a threefold is a smooth quintic in \mathbb{P}^4 . There are however much more examples, making the general theory of Calabi–Yau threefolds very complicated. One of the reasons is that even the simplest example (for instance the quintic) may admit many degenerations whose resolution will again be a Calabi–Yau threefold. Such an operation is called a geometric transition. Restricting the singularities of the degenerate variety to be nodes, one studies so-called conifold transitions. In [2], Strominger gave an interpretation of conifold transitions in the context of black hole condensations. Nowadays a few more geometric transitions have found a good interpretation in terms of superstring theory (see [3, 4, 5]).

In [7] Gross used geometric transition to formulate a version of the so-called “Reid fantasy” [6] which doesn’t need to go beyond the algebraic category. He conjectured that any two Calabi–Yau threefolds might be connected by sequences of geometric transitions. In Physical terms this statement is called the Web conjecture, as it suggests that all Calabi–Yau threefolds form a giant web, whose connections are geometric transitions.

So far it has been proven that all Calabi–Yau threefolds which are complete intersections in products of projective spaces are connected using geometric transitions (see [9]). Moreover, a large class of hypersurfaces in weighted projective four-space has also been connected to the web (see [10, 11]). In the latter context the candidates for the transitions are constructed, very naturally, by intersecting the polytopes defining the Calabi–Yau varieties, and checking whether the intersection is also a reflexive polytope. This construction in geometrical terms may be interpreted as a composition of projections of the degenerate Calabi–Yau threefold completed by a smoothing.

The aim of this paper is to study geometric transitions related with projections (or equivalently but more adequately in our context with unprojections). More precisely, under some additional assumptions, a projection will induce a pair of geometric transitions which we

shall call, to avoid confusion, a geometric bitransition. This class of bitransitions, thanks to its naturality, may be much better understood, and hence gives us a new variety of tools that may be used for explicit descriptions of examples of Calabi–Yau threefolds. The theory of unprojections has already proved its efficiency in classical algebraic geometry. Thanks to works of Reid and Papadakis [12, 13, 14, 15, 16], unprojections became a powerful tool to describe and construct new varieties. They are used in descriptions of singular K3 surfaces and Fano threefolds. In this paper we would like to show how these tools work in the context of Calabi–Yau threefolds. Differently from the approach to K3 surfaces or Fano 3 folds, we will not be interested in the singular varieties arising by unprojection, but in the families of smooth Calabi–Yau threefolds that might degenerate to such. The construction shall give rise to geometric bitransitions between these families. Throughout the paper we study examples of such given geometric bitransitions. In this way we find connections (consisting of an even number of geometric transitions) between many known families of Calabi–Yau threefolds with Picard number one. In particular we connect to the web the 5 families of Calabi–Yau threefolds introduced by Borcea [17] and described as complete intersections in homogeneous spaces.

The main advantage of our constructions is that at least in low codimension they are well understood in terms of equations. Thanks to that, for some known examples of Calabi–Yau threefolds we find descriptions in terms of equations in weighted projective space. This fact enables us to understand much better the geometry of those examples of Calabi–Yau varieties, which were known only as smoothings of some degenerate varieties.

Moreover, in two of the cases the construction leads us to new Calabi–Yau threefolds. They are described by the vanishing of Pfaffians of some 5×5 matrix with polynomial entries in some weighted projective spaces. These two families, together with a few others explicitly described in this paper, have conjectured Picard-Fuchs equations of their mirror (see [18]). We hope one might use the results of this paper to find these mirrors. There are at least two possibilities to proceed. The first is to use directly the constructed geometric transitions to find candidates for the mirrors. This idea is based on the conjecture by Morrisson (see [19]) which states that any geometric transition should admit a mirror transition going in the opposite direction. The second would be based on using the explicit descriptions found to proceed as in [20]. In most cases one could also try to mimic the ideas of [21] and [22]. The two new examples are additionally interesting because the conjectured Picard-Fuchs equations of their mirrors admit non-integral elliptic instanton numbers.

2. PRELIMINARIES

Although as announced in the introduction throughout the paper the term Calabi–Yau threefold shall concern only smooth projective manifold, for clarity of presentation we shall use also the term singular Calabi–Yau threefold as follows.

Definition 2.1. A normal Gorenstein variety X is called a singular Calabi–Yau threefold if it has trivial dualizing sheaf and satisfy $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Geometric transitions studied in this paper are defined as follows.

Definition 2.2. A geometric transition from a Calabi–Yau threefold X to a Calabi–Yau threefold Y is a pair consisting of a birational morphism $f : X \rightarrow Z$ and a flat family over a disc with central fiber Z and some other fiber Y . Where Z is a possibly (preferably) singular

Calabi–Yau threefold. In this context the latter deformation will be called a smoothing of Z or a degeneration of Y depending on the direction from which we look.

Definition 2.3. Two families \mathcal{X} , \mathcal{Y} of Calabi–Yau threefolds are connected by a geometric transition, if there exist smooth elements $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ that are connected by a geometric transition.

Example 2.4. A standard example is the following. Let Z be a general quintic containing a plane. It is a degeneration of a smooth quintic Y which has 16 nodes. The blow up X of Z in the plane is a smooth Calabi–Yau threefold. Then the considered blow up together with the deformation is a geometric transition connecting X and Y .

Definition 2.5. We shall say that two Calabi–Yau threefolds are connected by a geometric bitransition if there exists a Calabi–Yau threefold Z such that there are geometric transitions both from Z to X and from Z to Y .

Example 2.6. A standard example of bitransition is taken from [7], and is the following. Consider a quintic with a triple point. Blowing up this triple point is a resolution of singularities with an exceptional divisor a del Pezzo surface of degree 3. The second extremal ray of the obtained variety Z defines a map on a double octic threefold contracting 61 lines to nodes. In this way we connect by a geometric bitransition the family of quintic Calabi–Yau threefolds with the family of double octic Calabi–Yau threefolds (compare with case 15 in the Table of subsection 3.1).

Remark 2.7. Observe that with our definition two families of Calabi–Yau threefolds with Picard number one cannot be connected by one geometric transition. This is because smooth varieties with Picard number one admit no contraction morphisms. That is why we introduced the notion of bitransition as being the most natural way to connect two Calabi–Yau threefold of Picard number one. One could alternatively consider a weaker definition of geometric transition without assuming f to be a morphism. Then two Calabi–Yau threefolds with Picard number one could a priori be connected by such transitions, but unfortunately no such example is known to the author.

For a detailed introduction to the theory of geometric transitions see [1]. Let us now recall some basic facts about Kustin–Miller unprojections (for more details see [12, 14]). An unprojection is a birational map which is inverse to some projection. We shall consider unprojections arising in the following construction.

Let $D \subset X \subset \mathbb{P}^n$ be two projectively Gorenstein varieties such that D has codimension 1 in X . Let $\omega_X = \mathcal{O}_X(k_X)$ and $\omega_D = \mathcal{O}_D(k_D)$ be the dualizing sheaves of X and D . Assume that $k_X > k_D$. Then by [12, Thm 5.2] there exists a section $s \in \mathcal{O}_X(k_X - k_D)$ with poles along D which defines a birational map:

$$\varphi : X \dashrightarrow Y \subset \mathbb{P}^n[s] = \text{Proj } \mathbb{C}[x_0, \dots, x_n, s]$$

contracting D to the point $P_s := (0 : \dots : 0 : 1)$. Moreover Y is also projectively Gorenstein. The inverse of φ is the projection of Y from the point P_s .

Definition 2.8. The map φ obtained above will be called a Kustin Miller unprojection with exceptional divisor D and unprojected variety X .

Remark 2.9. The above construction doesn't need the ambient space \mathbb{P}^n . In fact it is often performed without any ambient space. In our context it will be convenient to have an ambient space $T = \text{Proj } R$ and then $Y \subset \text{Cone}(T) := \text{Proj } R[s]$

Remark 2.10. Observe that the projection inverse to φ always factorizes through the blow-up Z of P_s and a contraction morphism.

Example 2.11. The standard example of Kustin–Miller unprojection is based on the famous Reid $Ax - By$ trick. Let us recall it here following [12, sec.2]. Let $T = \mathbb{P}^n(a_0, \dots, a_n)$ with coordinate system $x = (x_0, \dots, x_n)$. Let $D = \{x \in T \mid A(x) = B(x) = 0\}$ for some homogeneous polynomials A and B of degrees d_1 and d_2 , and let X be a hypersurface containing D . Then $X = \{x \in T \mid A(x)C(x) - B(x)D(x) = 0\}$ with C and D homogeneous polynomials of degrees k_1, k_2 such that $k_1 + d_1 = k_2 + d_2$. Assume that $d_1 > k_2$. Then we can define $Y = \{x \in \text{Proj}(\mathbb{C}[x_0, \dots, x_n, s]) \mid C(x)s = A(x), D(x)s = B(x)\}$. Observe that Y contains the point $P_s = (0 : \dots : 0 : 1)$, and that the image of $Y \setminus \{P_s\}$ by the projection from P_s contains $X \setminus D$.

In this paper we shall consider the following construction based on Kustin–Miller unprojections of del Pezzo surfaces in Calabi–Yau threefolds. Take a del Pezzo surface D anti-canonically embedded in a (weighted) projective space, i.e. in such a way that the restrictions of the hyperplanes from the projective space is the complete anti-canonical system (D may be contained in some hyperplane). Consider moreover a proper flat family of Calabi–Yau threefolds with Picard number one embedded in the same projective space by the generators of their Picard groups. Assume that the elements of this family are projectively Gorenstein. Suppose that in the considered family we find a singular Calabi–Yau threefold X containing the surface D . As both D and X are projectively Gorenstein, by Kustin–Miller there is a rational section $s \in \mathcal{O}_X(1)$ with poles along D . The closure of the graph of this section (in the cone $\text{Cone}(T)$) is a variety Y birational to X , and which is a singular Calabi–Yau threefold with isolated singularity isomorphic to a cone over the anti-canonically embedded del Pezzo surface D . For most del Pezzo surfaces (except for the Hirzebruch surface \mathbb{F}_1) under some additional assumptions (see [7, 8]) such singular variety Y can be smoothed to a Calabi–Yau threefold. Moreover, the smoothing may then be performed in the same space in which the singular variety is embedded. In fact, in our context we won't need the existence of the smoothing as we will have an explicit description of the singular variety leading us to a natural explicit smoothing. As X can also be smoothed by assumption we obtain in this way a data consisting of a degeneration followed by a birational map and a smoothing. Moreover, our birational map can be factorized as in Remark 2.10. Now provided that Z is a Calabi–Yau threefold we obtain a geometric bitransition between two families of Calabi–Yau threefolds.

In order to use the above construction to find explicit examples of geometric transitions, we first need to find a suitable embedding of a del Pezzo surface into a singular Calabi–Yau threefold admitting a smoothing. Then, after having performed the unprojection, we need to find an explicit smoothing of the obtained singular variety. To complete the picture properly, we shall also need a crepant resolution of each of the considered singular varieties.

One can use the following lemma to construct a table of candidates for the considered construction in terms of basic invariants. Assume that we have two families \mathcal{X} and \mathcal{Y} of embedded Calabi–Yau threefolds connected by an unprojection in the above sense. Let X

and Y be general elements of these families and H_X and H_Y their respective hyperplane sections. Let $\pi: X_0 \rightarrow Y_0$ be the unprojection connecting the families. It is a birational map between two singular Calabi–Yau threefolds, $X_0 \in \mathcal{X}$ and $Y_0 \in \mathcal{Y}$ contracting a Del Pezzo surface D of degree d .

Lemma 2.12. *Under the above assumptions the invariants of X and Y are related by the following formulas:*

- (i) $\dim(|H_Y|) = \dim(|H_X|) + 1$
- (ii) $H_Y^3 = H_X^3 - d$
- (iii) $c_2(Y).H_Y = c_2(X).H_X - 12 + 2d$

Proof. Item (i) follows directly from the assumption. Item (ii) follows from the fact that the preimage of a general linear section of X_0 by π^{-1} is a linear section of Y_0 passing through the center of the projection π^{-1} and that the degree is equivariant under flat deformations. Item (iii) follows from the Riemann–Roch theorem for threefolds. \square

Remark 2.13. Observe that the invariants H^3 and $c_2.H$ are enough to determine whether there exists a flat deformation between two Calabi–Yau threefolds (see [23, Thm. 7.4]).

Remark 2.14. The constructed table of candidates should be considered as giving us only hints about what examples to look for. We will see however that it is very incomplete for two reasons. The first is that: even if a pair of families appears to be a good candidate basing only on the invariants, it does not necessarily mean that we can directly construct a suitable unprojection between them. The second is that the list of Calabi–Yau threefolds with Picard group of rank one is far to be complete. Hence even if a Calabi–Yau threefold appears not to admit any unprojection that connects it to other families using geometric transitions, it does not mean that there is no connection at all which uses these constructions. This means only that we need an intermediate family that is not included in the table. Moreover, as by unprojection we do not always end up in Calabi–Yau threefolds with Picard group of rank one, this intermediate family needs not to be of Picard number one.

The constructions involving unprojections are very convenient tools to describe already known varieties in terms of equations. For instance the following proposition shows that most constructions contained in [24, 25, 26] are in fact unprojections in the sense studied in this paper.

Proposition 2.15. *Let $X \subset \mathbb{P}^n$ be a nodal Calabi–Yau threefold containing a del Pezzo surface D in its anti-canonical embedding. Let H be the hyperplane section of X . Let \tilde{X} be a small resolution of X arising by blowing up D and flopping the obtained lines. Then the morphism given by the system $|\tilde{D} + \tilde{H}|$ on \tilde{X} is the composition of the blowing down with the unprojection map contracting D .*

Proof. Observe that the morphism $\varphi_{|\tilde{D} + \tilde{H}|}$ given by the system $|\tilde{D} + \tilde{H}|$ on \tilde{X} contracts \tilde{D} . Denote the image of D by p . Let us consider the composition of the morphism given by $|\tilde{D} + \tilde{H}|$ on \tilde{X} with the projection from p . This rational morphism is given outside \tilde{D} by the subsystem of $|\tilde{D} + \tilde{H}|$ given by divisors with D as component. This implies that the considered composition is given outside D by the system \tilde{H} , i.e. it is the blowing down of the small resolution. We hence obtain the equality from the assertion outside D which implies equality everywhere. \square

3. CLASSICAL UNPROJECTIONS

The classical examples of unprojections for which the result is well described using equations concern the situation where the unprojected variety (in our case D) is a complete intersection of low codimension.

3.1. codimension 2. Del Pezzo surfaces which are a codimension 2 complete intersections in some weighted projective space can be naturally embedded into singular Calabi–Yau threefolds which are hypersurfaces in the same space. This is done in the following way. Assume that D is given by equations q_1, q_2 of (weighted) degrees d_1, d_2 in some space T . Let \mathcal{X} be the family of hypersurfaces of degree $d > \max\{d_1, d_2\}$ in T . Then the variety X_0 given by the equation $q_1 a_1 + q_2 a_2 = 0$ is an element of \mathcal{X} that contains D for any a_1, a_2 of degrees $d - d_1, d - d_2$ respectively. We are then in position of Example 2.11. Hence the Kustin–Miller unprojection corresponding to our data leads to a codimension 2 complete intersection Y_0 in a cone over our starting space. As we observed in Remark 2.10 the inverse birational map factorizes through the blow up Z of the vertex P of the cone and a birational contraction morphism. In the considered case for general X_0 the contraction morphism contracts proper transforms of lines given by $q_1(x) = a_1(x) = q_2(x) = a_2(x) = 0$ in $\text{Cone}(T)$ to nodes given by the same equations in T . Now Z is a crepant resolution of both X_0 and Y_0 . The first being the blow up of a singularity with tangent cone a del Pezzo surface the latter a small resolution of nodes. It follows that Z is a Calabi–Yau threefold (in particular it is smooth) connected by a geometric transition to any smooth representative of \mathcal{X} and any smoothing of Y_0 . In each of the cases below the smoothing of Y_0 is explicit and hence the obtained pair of geometric transitions is an explicit geometric bitransition related to a Kustin–Miller unprojection.

T	del Pezzo	Calabi–Yau	result of unprojection
$T_6 \subset \mathbb{P}(1^2, 2^2, 3^2)$	$D_{2,3,6} \subset \mathbb{P}(1^2, 2^2, 3^2)$	$X_{6,6} \subset \mathbb{P}(1^2, 2^2, 3^2)$	$Y_{3,4,6} \subset \mathbb{P}(1^3, 2^2, 3^2)$
$T_6 \subset \mathbb{P}(1^3, 2^2, 3)$	$D_{1,2,6} \subset \mathbb{P}(1^3, 2^2, 3)$	$X_{4,6} \subset \mathbb{P}(1^3, 2^2, 3)$	$Y_{2,3,6} \subset \mathbb{P}(1^4, 2^2, 3)$
$T_6 \subset \mathbb{P}(1^4, 2, 3)$	$D_{1,1,6} \subset \mathbb{P}(1^4, 2, 3)$	$X_{3,6} \subset \mathbb{P}(1^4, 2, 3)$	$Y_{2,2,6} \subset \mathbb{P}(1^5, 2, 3)$
$T_4 \subset \mathbb{P}(1^3, 2^2, 3)$	$D_{4,2,3} \subset \mathbb{P}(1^3, 2^2, 3)$	$X_{4,6} \subset \mathbb{P}(1^3, 2^2, 3)$	$Y_{4,4,3} \subset \mathbb{P}(1^4, 2^2, 3)$
$T_4 \subset \mathbb{P}(1^4, 2^2)$	$D_{1,2,4} \subset \mathbb{P}(1^4, 2^2)$	$X_{4,4} \subset \mathbb{P}(1^4, 2^2)$	$Y_{2,3,4} \subset \mathbb{P}(1^5, 2^2)$
$T_4 \subset \mathbb{P}(1^5, 2)$	$D_{1,1,4} \subset \mathbb{P}(1^5, 2)$	$X_{3,4} \subset \mathbb{P}(1^5, 2)$	$Y_{2,2,4} \subset \mathbb{P}(1^6, 2)$
$\mathbb{P}(1^4, 2)$	$D_{2,3} \subset \mathbb{P}(1^4, 2)$	$X_6 \subset \mathbb{P}(1^4, 2)$	$Y_{3,4} \subset \mathbb{P}(1^5, 2)$
$T_3 \subset \mathbb{P}(1^5, 2)$	$D_{1,2,3} \subset \mathbb{P}(1^5, 2)$	$X_{3,4} \subset \mathbb{P}(1^5, 2)$	$Y_{2,3,3} \subset \mathbb{P}(1^6, 2)$
$T_3 \subset \mathbb{P}^5$	$D_{1,1,3} \subset \mathbb{P}^5$	$X_{3,3} \subset \mathbb{P}^5$	$Y_{2,2,3} \subset \mathbb{P}^6$
$T_2 \subset \mathbb{P}(1^5, 3)$	$D_{2,2,3} \subset \mathbb{P}(1^5, 3)$	$X_{2,6} \subset \mathbb{P}(1^5, 3)$	$Y_{2,3,4} \subset \mathbb{P}(1^6, 3)$
$T_2 \subset \mathbb{P}^5$	$D_{1,2,2} \subset \mathbb{P}^5$	$X_{2,4} \subset \mathbb{P}^5$	$Y_{2,2,3} \subset \mathbb{P}^6$
$T_{2,2} \subset \mathbb{P}^6$	$D_{1,1,2,2} \subset \mathbb{P}^6$	$X_{2,2,3} \subset \mathbb{P}^6$	$Y_{2,2,2,2} \subset \mathbb{P}^7$
$Pf \subset \mathbb{P}(1^7, 2)$	$D_{1,2} \cap Pf \subset \mathbb{P}(1^7, 2)$	$X_4 \cap Pf \subset \mathbb{P}(1^7, 2)$	$Y_{2,3} \cap Pf \subset \mathbb{P}(1^8, 2)$
$Pf \subset \mathbb{P}^7$	$D_{1,1} \cap Pf \subset \mathbb{P}^7$	$X_3 \cap Pf \subset \mathbb{P}^7$	$Y_{2,2} \cap Pf \subset \mathbb{P}^8$
$\mathbb{P}(1^4, 4)$	$D_{3,4} \subset \mathbb{P}(1^4, 4)$	$X_8 \subset \mathbb{P}(1^4, 4)$	$Y_{4,5} \subset \mathbb{P}(1^5, 4)$
$\mathbb{P}(1^3, 2, 5)$	$D_{4,5} \subset \mathbb{P}(1^3, 2, 5)$	$X_{10} \subset \mathbb{P}(1^3, 2, 5)$	$Y_{5,6} \subset \mathbb{P}(1^4, 2, 5)$
$T_6 \subset \mathbb{P}(1^4, 2, 3)$	$D_{1,1,6} \subset \mathbb{P}(1^4, 2, 3)$	$X_{3,6} \subset \mathbb{P}(1^4, 2, 3)$	$Y_{2,2,6} \subset \mathbb{P}(1^5, 2, 3)$
$T_2 \subset \mathbb{P}(1^5, 3)$	$D_{2,2,3} \subset \mathbb{P}(1^5, 3)$	$X_{2,6} \subset \mathbb{P}(1^5, 3)$	$Y_{2,4,3} \subset \mathbb{P}(1^6, 3)$
$T_4 \subset \mathbb{P}(1^4, 2)$	$D_{2,3} \subset \mathbb{P}(1^4, 2)$	$X_6 \subset \mathbb{P}(1^4, 2)$	$Y_{3,4} \subset \mathbb{P}(1^5, 2)$

T	del Pezzo	Calabi–Yau	result of unprojection
\mathbb{P}^4	$D_{3,1} \subset \mathbb{P}^4$	$X_5 \subset \mathbb{P}^4$	$Y_{2,4} \subset \mathbb{P}^5$
\mathbb{P}^4	$D_{2,2} \subset \mathbb{P}^4$	$X_5 \subset \mathbb{P}^4$	$Y_{3,3} \subset \mathbb{P}^5$

The notation Pf in the table stands for the variety given by 4×4 Pfaffians of a 5×5 matrix with general linear entries in the appropriate space (it describes either a linear section or a cone over the Grassmannian $G(2,5)$ in it's Plücker embedding). Here all the varieties obtained as results of the unprojections in the table are well known to admit smoothings, as the general members with given description are known to be classical examples of smooth Calabi–Yau threefolds.

Remark 3.1. One can also use methods of [25] or direct computation with Macaulay 2 [30], that in all above cases the unprojection factorizes into a small resolution of nodes and a primitive contraction of the del Pezzo surface as described in Lemma 2.15. The latter factorization is the same as the one from Remark 2.10.

3.2. codimension 3. Del Pezzo surfaces which are a codimension 3 complete intersections in some weighted projective space can be naturally embedded into appropriate singular Calabi–Yau threefolds which are codimension 2 complete intersections. By [14] the unprojection corresponding to such data leads us to a variety given by 4×4 Pfaffians of a 5×5 matrix with entries of appropriate weight. More precisely let D be given by the equations q_1, q_2, q_3 of degrees d_1, d_2, d_3 respectively. Let \mathcal{X} be a family of Calabi–Yau threefolds given as complete intersections of hypersurfaces of degrees e_1, e_2 , where $\min\{e_1, e_2\} > \max\{d_1, d_2, d_3\}$. Let X_0 be given by equations $q_1 a_1 + q_2 a_2 + q_3 a_3 = 0$, $q_1 b_1 + q_2 b_2 + q_3 b_3 = 0$, where for all $i \in \{1, 3\}$ we have a_i, b_i are generic of degree $e_1 - d_i, e_2 - d_i$ respectively. The variety Y_0 given in the cone over T by the Pfaffians of the matrix

$$\begin{pmatrix} 0 & t & a_1 & a_2 & a_3 \\ -t & 0 & b_1 & b_2 & b_3 \\ -a_1 & -b_1 & 0 & q_3 & -q_2 \\ -a_2 & -b_2 & a_1 & 0 & q_1 \\ -a_3 & -b_3 & a_1 & a_2 & 0 \end{pmatrix}$$

is then a variety whose projection from $t = 1$ (all other coordinates zero) is X_0 and such that the exceptional divisor of the projection is given by $q_1 = 0, q_2 = 0, q_3 = 0$. Here in the same way as before the blow up Z of the singular point is a crepant resolution of Y_0 and the morphism induced on Z to X_0 is a small contraction to nodes of X_0 . We hence obtain an explicit geometric bitransition in each of the following examples.

	T	del Pezzo	Calabi–Yau	result of unprojection
1	$T_4 \subset \mathbb{P}(1^6, 2)$	$D_{1,1,1,4} \subset \mathbb{P}(1^6, 2)$	$X_{2,2,4} \subset \mathbb{P}(1^6, 2)$	$Y_4 \cap Pf \subset \mathbb{P}(1^7, 2)$
2	$T_3 \subset \mathbb{P}^6$	$D_{1,1,1,3} \subset \mathbb{P}^6$	$X_{2,2,3} \subset \mathbb{P}^6$	$Y_3 \cap Pf \subset \mathbb{P}^7$
3	\mathbb{P}^5	$D_{1,2,2} \subset \mathbb{P}^5$	$X_{3,3} \subset \mathbb{P}^5$	$WPf(1, 2) \subset \mathbb{P}^6$
4	$T_{2,2} \subset \mathbb{P}^7$	$D_{1,1,1,2,2} \subset \mathbb{P}^7$	$X_{2,2,2,2} \subset \mathbb{P}^7$	$Y_{2,2} \cap Pf \subset \mathbb{P}^8$
5	$Pf \subset \mathbb{P}^8$	$D_{1,1,1} \cap Pf \subset \mathbb{P}^8$	$X_{2,2} \cap Pf \subset \mathbb{P}^8$	$Pf \cap Pf \subset \mathbb{P}^9$
6	$\mathbb{P}(1^5, 2)$	$D_{2,2,2} \subset \mathbb{P}(1^5, 2)$	$X_{3,4} \subset \mathbb{P}(1^5, 2)$	$WPf(1, 2) \subset \mathbb{P}(1^6, 2)$
7	$\mathbb{P}(1^4, 2^2)$	$D_{3,2,2} \subset \mathbb{P}(1^4, 2^2)$	$X_{4,4} \subset \mathbb{P}(1^4, 2^2)$	$WPf(1, 2, 3) \subset \mathbb{P}(1^5, 2^2)$

Let us look more precisely at the examples from the table. The result of the unprojections in the examples 2 and 4 are well known families of Calabi–Yau threefolds. These are complete intersections in the Grassmannian $G(2, 5)$. The example number 3 is also well known and described by 4×4 Pfaffians of a 5×5 matrix with one row of quadrics and remaining entries linear. The example 1 is the double cover of the Fano threefold B_5 obtained as a codimension 3 linear section of $G(2, 5)$ branched over the intersection of this Fano variety with a quartic (see [17]). The example number 5 was found in [25], it is a Calabi–Yau threefold with Picard number one. We find it’s explicit description as a complete intersection of two Grassmannians embedded in \mathbb{P}^9 by different Plücker embeddings. The examples 5 and 6 are new examples of Calabi–Yau threefolds with Picard number one. Calabi–Yau threefolds with such invariants have a predicted Picard-Fuchs equation of their mirror in [18].

Example 3.2. The example 6 is described by Pfaffians of a generic antisymmetric 5×5 matrix with entries of the following degrees:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 \\ & & 2 & 2 \\ & & & 2 \end{pmatrix}$$

in the weighted projective space $\mathbb{P}(1^6, 2)$. To prove that these are smooth varieties we use the computer algebra system Macaulay 2 on a specific example. The hodge numbers can be computed directly from the above description. We can also proceed as in [25, Thm 2.2]. We need only to observe that the general X_4 containing a del Pezzo surface $D_{2,2,2} \subset \mathbb{P}(1^5, 2)$ is smooth and the generic such $X_{3,4} \subset \mathbb{P}(1^5, 2)$ has 28 nodes. We obtain that $h^{1,1}(Y) = 1$ and $h^{1,2} = 60$. Hence the Euler characteristic $\chi(Y) = -116$. The remaining invariants follow from Lemma 2.12 and are $H_Y^3 = 10$, $\dim(|H_Y|) = 6$ and $c_2(Y).H = 52$.

Example 3.3. The example 7 is described by Pfaffians of a generic antisymmetric 5×5 matrix with entries of the following degrees:

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}$$

in the weighted projective space $\mathbb{P}(1^5, 2^2)$. Observe that it is a section of a cone (with vertex a line) over a weighted Grassmannian $wG(2, 5) \subset \mathbb{P}(1^3, 2^6, 3)$ with weights $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ (see [31]) by a generic cubic and 4 generic quadrics. Analogously as before we prove that these are Calabi–Yau threefolds with Picard number one, Euler characteristic $\chi(Y) = -120$, $h^{1,2} = 62$, $H_Y^3 = 7$, $\dim(|H_Y|) = 5$ and $c_2(Y).H = 46$.

Remark 3.4. As we are given explicit descriptions of examples 5, 6, 7 it would be interesting to check whether the Picard-Fuchs equations of their mirrors are indeed those found in [18]. In fact examples 1 and 3 have also only conjectured corresponding equations. One might try to proceed as in [20] using above geometric transitions instead of conifolds. What is additionally interesting is that the Calabi–Yau equations that by [18, Table 1.] should correspond to examples 6 and 7 (they produce the same invariants including the Euler Characteristic) produce some non-integral elliptic instantons n_d^1 . This would contradict the conjecture concerning their integrality.

Example 3.5. We might have also considered the following unprojection triple. $T_6 \subset \mathbb{P}(1^5, 2, 3)$, $D_{1,1,1,6} \subset \mathbb{P}(1^5, 2, 3)$, $X_{2,2,6} \subset \mathbb{P}(1^5, 2, 3)$. The result of such an unprojection should be of the form $Y_6 \cap Pf \subset \mathbb{P}(1^6, 2, 3)$. We however observe that such a variety is always singular and it is not clear whether it admits a smoothing.

3.3. Tom and Jerry. Del Pezzo surfaces which are codimension 4 complete intersections can naturally be embedded into Pfaffian threefolds in two ways called Tom and Jerry. We find two examples of Calabi–Yau varieties that might fit to this construction. Both are described by Pfaffians of matrices of linear forms in some spaces T . Then, the del Pezzo surfaces that we might try to unproject are given as linear sections of T as shown in the table below.

T	del Pezzo	Calabi–Yau	result of Tom	result of Jerry
$T_{2,2} \subset \mathbb{P}^8$	$D_{1,1,1,1,2,2} \subset \mathbb{P}^8$	$Pf \cap X_{2,2} \subset \mathbb{P}^8$	$Tom \cap Y_{2,2} \subset \mathbb{P}^9$	$Jerry \cap Y_{2,2} \subset \mathbb{P}^9$
$Pf \subset \mathbb{P}^9$	$D_{1,1,1,1} \cap Pf \subset \mathbb{P}^9$	$Pf \cap Pf \subset \mathbb{P}^9$	$Tom \cap Pf \subset \mathbb{P}^{10}$	$Jerry \cap Pf \subset \mathbb{P}^{10}$

Both cases are in fact sections of standard constructions called original Tom and original Jerry by different spaces T . More precisely D is given as a complete linear section of codimension 4 in T (write l_1, \dots, l_4 for its defining linear forms). The Calabi–Yau varieties from the family are described by 4×4 Pfaffians of linear forms of a 5×5 matrix with linear entries. Consider the varieties Y_1 given by 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & h_1 & h_2 & h_3 & h_4 \\ -h_1 & 0 & 0 & l_1 & l_2 \\ -h_2 & 0 & 0 & l_3 & l_4 \\ -h_3 & -l_1 & -l_3 & 0 & 0 \\ -h_4 & -l_2 & -l_4 & 0 & 0 \end{pmatrix}$$

and Y_2 given by 4×4 Pfaffians of the matrix

$$\begin{pmatrix} 0 & l_1 & l_2 & l_3 & 0 \\ -l_1 & 0 & 0 & l_3 & l_4 \\ -l_2 & 0 & 0 & h_1 & h_2 \\ -l_3 & -l_3 & -h_1 & 0 & h_3 \\ 0 & -l_4 & -h_2 & -h_3 & 0 \end{pmatrix}.$$

Both these varieties contain D . One checks easily that the result of these standard unprojections are cones over the Segre embeddings of $\mathbb{P}^2 \times \mathbb{P}^2$ for Tom and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ for Jerry. Hence the results of the unprojections are intersections of these cones with the corresponding space T . It follows that in each case the blow up of the singularity of the variety Y is a crepant resolution with exceptional divisor a del Pezzo surface of degree 6. Observe that although these varieties admit explicit smoothings, the smoothed varieties do not have Picard number 1.

4. CODIMENSION 3

In all above examples we restricted to the case where the del Pezzo surface D is given as a complete intersection in the space T . However, for the Kustin–Miller unprojection to exist

it is only needed that both D and X are Gorenstein. Below we deal with the general case in codimension 3. We have the following proposition.

Proposition 4.1. *Let D be given by the $n - 1 \times n - 1$ Pfaffians of an $n \times n$ antisymmetric matrix M of linear forms in some projective space. Let X be a complete intersection of two hypersurfaces h_1, h_2 containing D . Then there is a variety Y given by $n + 1 \times n + 1$ minors of an $n + 2 \times n + 2$ antisymmetric matrix of linear forms, which is the unprojection of D in X .*

Proof. Let us denote the Pfaffians of M by f_1, \dots, f_n . We can then write $h_1 = a_1 f_1 + \dots + a_n f_n$, $h_2 = b_1 f_1 + \dots + b_n f_n$. Let t be the additional variable in the cone over T . Consider the matrix

$$N = \begin{pmatrix} 0 & t & a_1 & \dots & a_n \\ -t & 0 & b_1 & \dots & b_n \\ a_1 & b_1 & & & \\ \vdots & \vdots & & \mathbf{M} & \\ a_n & b_n & & & \end{pmatrix}$$

The $n + 1 \times n + 1$ Pfaffians of N define a variety Y whose projection from $t = 1$ (all coordinates in T being 0) is the variety X and the exceptional divisor of the projection is D . \square

Example 4.2. By the above we recover the construction from [24] of the geometric bi-transition of a del Pezzo of degree 5 in the Calabi–Yau threefold $X_{3,3} \subset \mathbb{P}^5$, obtaining the Calabi–Yau variety given by 6×6 Pfaffians of a generic 7×7 antisymmetric matrix.

Remark 4.3. There are also other examples where the unprojection may be performed explicitly in a general context. For instance the constructions contained in [26] are Kustin–Miller unprojections with explicit descriptions of the resulting variety. All these may be performed in pure algebraic terms and hence be extended to a more general context. For instance one may formulate a cascades of unprojections involving such determinantal varieties of any size with additional weights some of which might also give constructions of Calabi–Yau threefolds.

5. LINKAGE

Unfortunately in higher codimension there isn't any general description of the unprojection. However, if D is a Del Pezzo surface embedded into a complete intersection nodal Calabi–Yau threefold X we can construct a linkage between a general hyperplane section of the variety Y and the del Pezzo surface D .

Indeed, we see that taking a section of X given by a general hypersurface K of degree k containing D . We obtain $X \cap K = D \cup S$ for some surface $S \in |kH - D|$. Next, taking a generic hypersurface L of degree $K + 1$ containing S we get $X \cap L = D \cup G$, where G is a surface from the system $|D + H|$. Hence by Proposition 2.15 (or directly from the existence of Kustin–Miller unprojection) it is isomorphic to a general hyperplane section of Y .

The problem with the above approach is that in this way we find a description of the resolution of the singular variety which is the result of the unprojection. But a priori there is no reason for the smoothed variety to have the same resolution. In the examples we dealt with so far the resolution of the smoothing was always of the same form as the resolution of the result of the unprojection. This did not follow however from a general principle but

from the simplicity of the construction which implied that we were always given a natural family with given resolution and whose generic member is smooth.

5.1. Unprojections of Del Pezzo surfaces of degree 6. The case D is a del Pezzo surface of degree 6 is somewhat special in this context. This is because as it was proven in [24] the singular variety obtained by such an unprojection is expected to have two different smoothing families. We shall see that in this case although the smoothings are different, their projective resolutions both have the same Betti numbers. Indeed, let D be a del Pezzo surface of degree 6. It is a classical fact that the del Pezzo surface $D \subset \mathbb{P}^6$ has two different descriptions. The first sees it as a codimension two complete linear section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, whilst the second interprets it as a hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$. Now, instead of looking at the generic hyperplane sections of the constructed variety Y , we might consider our unprojection as a hyperplane section of a higher dimensional unprojection involving one of these Segre embedded products. In order to perform the above reasoning we need to extend our Calabi–Yau threefold X as a linear section of an appropriate Fano variety. This is easily done if the Calabi–Yau threefold is a complete intersection. We just consider the generic complete intersection of the same kind in the higher-dimensional space. We can hence perform two constructions. The first will be the unprojection of a Fano fivefold containing $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. The second will be the unprojection of a Fano fourfold containing $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$. We see Y in both pictures as a hyperplane section passing through the singular point. Now reasoning as in [25, Thms 2.1, 2.2] we prove that the results of the unprojections in both cases have isolated singularities. As the they are also irreducible their generic hyperplane sections are smooth and define two smoothing families. One smoothing of Y will be the general hyperplane section of the constructed fourfold, whilst the second smoothing will be a general codimension 2 linear section of the fivefold. The smoothing inducing different local pictures of the smoothed singularity have to be different. Let us perform computations on an explicit example.

Example 5.1. Consider the unprojection of D_6 in a Calabi–Yau threefold $X_{2,2,3} \subset \mathbb{P}^6$. We can proceed in two ways. The first is to take the variety $D = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ in it's Segre embedding and a generic variety of type $\bar{X}_{2,2,3} \subset \mathbb{P}^8$ containing D . By the above discussion the smoothing of the result of the unprojection of D_6 in $X_{2,2,3}$ is connected by a linkage in $\bar{X}_{2,2,3}$ with $D \cap H \subset \mathbb{P}^7$. In particular the resolution is described by a mapping cone construction applied twice. First, on the resolutions of $D \cap H$ and the Koszul complex of $X_{2,2,3} \cap H \cap Q$, next on the result of this and the Koszul complex of $X_{2,2,3} \cap H \cap C$ in the second step. The Betti diagram of the resolution is then as follows

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \\ 0 & 2 & 0 & 0 & 0 & \\ 0 & 8 & 18 & 8 & 0 & . \\ 0 & 0 & 0 & 2 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array}$$

The second possibility is to take the variety $D = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ with $\bar{X}_{2,2,3} \subset \mathbb{P}^7$ containing D . Again we construct a linkage and end up with a smoothing with resolution having the same Betti diagram.

Remark 5.2. In general it is hard to distinguish between the two smoothings of a Calabi–Yau threefold with a singularity which is a cone over the del Pezzo surface D_6 . The existence

of both smoothings is proved first in the local context and then extended to the global case using formal cohomology. Here thanks to limiting ourselves to unprojections we extend the local Picture to the global case as described above. Both smoothings appear as sections of projective varieties constructed as higher dimensional unprojections of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2 \times \mathbb{P}^2$.

Remark 5.3. We can perform a similar construction in other cases. It surely works for other del Pezzo surfaces, but it might also be successful when the Calabi–Yau threefolds are not complete intersections. In the latter cases the construction of the Fano fourfolds extending our Calabi–Yau threefolds is much more delicate. However, once we manage to construct the higher dimensional unprojection in such a way that it ends up in a variety with isolated singularities our smoothing will be obtained as a general section of the constructed variety. The linkage construction then, gives us a more or less explicit description of the smoothed variety by giving its resolution.

6. COMPLETE INTERSECTIONS IN HOMOGENEOUS SPACES

In this section we consider classical examples of Calabi–Yau threefolds which are complete intersections in homogeneous varieties (see [17]). We prove that we can construct the following cascade of unprojections with a del Pezzo surface of degree 4 at each step.

H^3	χ	$c_2 \cdot H$	$\dim H $	Description
4	-256	52	5	$X_{2,6} \subset \mathbb{P}(1, 1, 1, 1, 1, 3)$
8	-176	56	6	$X_{2,4} \subset \mathbb{P}^5$
12	-144	60	7	$X_{2,2,3} \subset \mathbb{P}^6$
16	-128	64	8	$X_{2,2,2,2} \subset \mathbb{P}^7$
20	-120	68	9	$X_{1,2,2} \subset G(2, 5)$
24	-116	72	10	$X_{1,1,1,1,1,1,2} \subset S_{10}$
28	-116	76	11	$X_{1,1,1,1,2} \subset G(2, 6)$
32	-116	80	12	$X_{1,1,2} \subset LG(3, 6)$
36	-120	84	13	$X_{1,2} \subset G_2$

Here $LG(3, 6)$ denotes the Lagrangian Grassmannian, S_{10} stands for the orthogonal Grassmannian parameterizing four-dimensional spaces in a quadric of maximal rank in \mathbb{P}^9 , and G_2 is the variety parameterizing isotropic 5-spaces of a general 4-form in \mathbb{C}^7 . The varieties $X_2 \subset G(2, 5)$, S_{10} , $G(2, 6)$, $LG(3, 6)$ and G_2 are commonly called Mukai varieties and denoted alternatively by M_6, \dots, M_{10} respectively.

Remark 6.1. Observe that we saw already three possibilities of unprojecting a del Pezzo surface of degree 4 in a degeneration of a Calabi–Yau threefold corresponding to the family $X_{2,2} \subset G(2, 5)$. These were Tom and Jerry and the middle part of the cascade. However, only the last one leads us to a Calabi–Yau threefold with Picard group of rank one.

Up to the case $D_4 \subset X_{1,2,2} \cap G(2, 5)$ the unprojections are standard examples discussed above. To proceed with the other cases we again increase the dimension of the pictures and use slightly more general theorems that can be found in [27, 28, 29]. It is proved there that a general nodal proper linear section of a Mukai variety M_g projected from its node is a proper linear section of M_{g-1} .

Having this we construct needed unprojections as sections of these higher dimensional unprojections by an appropriate number of hyperplanes and a quadric cone. We construct in this way the whole cascade. To obtain geometric bitransitions we need to prove that in each case there is an intermediate Calabi–Yau threefold Z with morphisms factorizing the unprojection. It will again just be the blow up of the singularity of the projected threefold. Its tangent cone is isomorphic to some generic complete intersection of two quadric cones, hence to a cone over the del Pezzo surface of degree 4. The blow up of this singularity is a crepant resolution and the projection factorizes through it and a contraction morphism, which can easily be checked to be a small contraction.

Remark 6.2. In the same way as in Remark 3.4, the mirrors of the three examples of degrees 24, 32, and 36 are not known, but have a conjectured Picard-Fuchs equation. One may try to construct the mirrors using the constructed geometric transitions.

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